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On the Heptic Non-Homogeneous Equation with Four Unknowns $xy(x+y) + zw^6 = 0$

M.A.Gopalan[*1], G.Sumathi[2], S.Vidhyalakshmi[3]

[*1,2,3] Department of Mathematics,Shrimati Indira Gandhi College,Trichy, India

mayilgopalan@gmail.com

Abstract

The heptic non-homogeneous equation with four unknowns represented by the diophantine equation $xy(x+y)+2zw^6=0$ is analyzed for its patterns of non-zero distinct integral solutions and two different methods of integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely, Pyramidal numbers, Pronic numbers, Star numbers, Gnomonic numbers, Centered Polygonal numbers, Centered Hexagonal Pyramidal numbers, fourth dimensional figurate numbers are exhibited.

Keywords: Heptic non-homogeneous equation , Pyramidal numbers, Pronic numbers, fourth dimensional figurate numbers.

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

$T_{m,n}$: Polygonal number of rank n with size m

S_n : Star number of rank n

PR_n : Pronic number of rank n

$Gnomic_n$: Gnomonic number of rank n

Ct_n^m : Centered polygonal number of rank n with size m.

$Cf_{3,n,6}$: Centered Hexogonal Pyramidal number of rank n

$Cf_{4,n,3}$: Fourth dimensional triangular figurate number of rank n

Introduction

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3].Particularly in [4,5] special equations of sixth degree with four and five unknowns are studied. In [6-8] heptic equations with three and five unknowns are analysed.This communication analyses a hextic equation with four unknowns given by $xy(x+y)+2zw^6=0$. Infinitely many non-zero integer quadtuples (x, y, z, w) satisfying the above equation are obtained. Various interesting properties among the values of x, y, z and w are presented.

Method of Analysis

The diophantine equation representing a non-homogeneous heptic equation is

$$xy(x+y)+2zw^6=0 \qquad (1)$$

To start with,it is observed that (1) is satisfied by the following integer quadruples (x, y, z, w):

$$(32b^4\alpha^6, -2b^2, 16b^4\alpha^6 - b^2, 2\alpha b)\ ,\ (8\alpha^4, -8\alpha^2, 4\alpha^2(\alpha^2-1), 2\alpha)$$

Introducing the linear transformations

$$x = u+v, y = u-v, z = u \qquad (2)$$

in (1),it leads to

$$v^2 = u^2 + w^6 \qquad (3)$$

(3) can be written as

$$v^2 = u^2 + (w^3)^2 \qquad (4)$$

Pattern:1

Note that (4) is similar to the well known Pythogorean equation.using the most cited solution of standard Pythogorean equation,we have

$$\left.\begin{aligned} u &= 2AB \\ v &= A^2 + B^2 \end{aligned}\right\} \qquad (5)$$

$$w^3 = A^2 - B^2 \quad \text{,where } A, B > 0 \qquad (6)$$

Note that (6) is satisfied by

$$\left.\begin{aligned} A &= m(m^2 - n^2) \\ B &= n(m^2 - n^2) \end{aligned}\right\} \qquad (7a)$$

$$w = (m^2 - n^2)\ ,\quad \text{where } m, n > 0 \qquad (7b)$$

and thus from (5),we have

$$\left.\begin{aligned} u &= 2mn(m^2 - n^2)^2 \\ v &= (m^2 - n^2)^2(m^2 + n^2) \end{aligned}\right\} \qquad (8)$$

Substituting (8) in (2), the corresponding values of x, y, z are given by

$$\left.\begin{aligned} x &= (m^2 - n^2)^2(m+n)^2) \\ y &= -(m^2 - n^2)^2(m-n)^2 \\ z &= 2mn(m^2 - n^2)^2 \end{aligned}\right\} \qquad (9)$$

Thus (9) and (7b) represents the non-zero distinct integer solution to (1).

Properties:

1.Each of the following is a nasty number

a. $3[x(2m,m) + y(2m,m)]$

b. $18[x(2m,m) + y(2m,m) + z(2m,m)]$

c. $3[(x(2m,m) + y(2m,m))(z(2m,m))]$

2. $x(m,1) + z(m,1) = (24Cf_{4,m,3} - 6Cf_{3,m,6} + S_m - 18t_{4,m})(Ct_{2,m} + 3\Pr_m - t_{4,m})$

3. $x(1,n) + y(1,n) - ((4Cf_{3,n,6})(2Gnomic_n + t_{12,n} - 4t_{4,n})) \equiv 0 (\bmod 4)$

4. $15(x(4m,m)w(4m,m))$ is a perfect square.

5. Each of the following is a cubic integer.

a. $400(x(3m,m) - y(3m,m) + w(3m,m) - 8t_{4,m})$

b. $\dfrac{2(x(1,n) + y(1,n) - 4\Pr_n + 4t_{4,n})}{(2Gnomic_n + t_{12,n} - 4t_{4,n})}$

6. $z(2m,m)w(2m,m)$ is a biquadratic integer.

Pattern:2

(6) is solved through a different process as shown below:

Take $w = p^2 - q^2$,where $p,q > 0$ (10)

in (6) and employing the method of factorization, it is written as the expression of double equations

$$A + B = (p+q)^3$$
$$A - B = (p-q)^3$$

which is satisfied by

$$A = p^3 + 3pq^2$$
$$B = q^3 + 3p^2q$$

From (5) and (2),we get

$$\left.\begin{aligned} x &= \left(2(p^3+3pq^2)(q^3+3p^2q)\right) + \left((p^3+3pq^2)^2 + (q^3+3p^2q)^2\right) \\ y &= \left(2(p^3+3pq^2)(q^3+3p^2q)\right) - \left((p^3+3pq^2)^2 + (q^3+3p^2q)^2\right) \\ z &= \left(2(p^3+3pq^2)(q^3+3p^2q)\right) \end{aligned}\right\} \quad \text{.....................} \quad (11)$$

Thus (5) ,(10) and (11) represent the non-zero integral solutions to (1).

Pattern:3

(4) is rewritten as

$$u^2 + (w^3)^2 = v^2 \times 1 \quad (12)$$

Assume $v = v(a,b) = a^2 + b^2,\ a,b > 0$ (13)

and write 1 as

$$1 = \frac{\left((m^2-n^2) + i2mn\right)\left((m^2-n^2) - i2mn\right)}{(m^2+n^2)^2}, \quad m,n > 0 \quad (14)$$

Substituting (13) and (14) in (12), employing the method of factorization,define

$$(u + iw^3) = \frac{(m^2 - n^2 + i2mn)(a+ib)^2}{(m^2+n^2)} \quad (15)$$

from which we get

$$u = (m^2+n^2)^3\left((m^2-n^2)(\overline{A}^2 - \overline{B}^2) - 4mn\overline{A}\overline{B}\right) \quad (16)$$
$$w^3 = (m^2+n^2)^3\left(2mn(\overline{A}^2 - \overline{B}^2) + 2\overline{A}\overline{B}(m^2-n^2)\right) \quad (17)$$

where $a = (m^2+n^2)\overline{A}, b = (m^2+n^2)\overline{B}$ (18)

choose $\overline{A}$ and $\overline{B}$ such that

$$2mn(\overline{A}^2 - \overline{B}^2) + 2\overline{A}\overline{B}(m^2-n^2) = \alpha^3 \quad (19)$$

and thus $w = (m^2+n^2)\alpha$ (20)

Treating (19) as a quadratic in $\overline{A}$ and solving for $\overline{A}$,we have

$$\overline{A} = \frac{\left(-\overline{B}(m^2-n^2) \pm \sqrt{\overline{B}^2(m^2+n^2)^2 + 2mn\alpha^3}\right)}{2mn} \quad (21)$$

This square root on the right hand side of the above equation is removed by replacing

$$R^2 = \overline{B}^2(m^2+n^2)^2 + 2mn\alpha^3 \quad (22)$$

Again replacing $\alpha, \overline{B}, R$ respectively by

$$\left.\begin{aligned} \alpha &= 2mn\overline{\alpha} \\ \overline{B} &= (2mn)^2 \overline{\overline{B}} \\ R &= (2mn)^2 \overline{R} \end{aligned}\right\} \qquad (23)$$

in the above equation, it becomes

$$\overline{R}^2 - (m^2+n^2)^2 \overline{\overline{B}}^2 = \overline{\alpha}^3 \qquad (24)$$

which is satisfied by

$$\overline{R} = \frac{\overline{\alpha}^2 + \overline{\alpha}}{2} \qquad (25)$$

$$\overline{\overline{B}} = \frac{\overline{\alpha}^2 - \overline{\alpha}}{2(m^2+n^2)} \qquad (26)$$

Again, taking $\overline{\alpha} = (m^2+n^2)\alpha^*$ (27)

we have

$$\overline{R} = \frac{(m^2+n^2)\alpha^*\left((m^2+n^2)\alpha^*+1\right)}{2} \qquad (28)$$

$$\overline{\overline{B}} = \frac{\alpha^*\left[(m^2+n^2)\alpha^*-1\right]}{2} \qquad (29)$$

From (21),(28) and (27) in (23),we have

$$\left.\begin{aligned} \overline{B} &= (2mn)^2 \frac{\alpha^*\left[(m^2+n^2)\alpha^*-1\right]}{2} \\ R &= (2mn)^2 \frac{(m^2+n^2)\alpha^*\left((m^2+n^2)\alpha^*+1\right)}{2} \\ \alpha &= (2mn)(m^2+n^2)\alpha^* \end{aligned}\right\} \qquad (30)$$

Substituting the above values of α and $\overline{B}$ in (21) and taking the positive value of the square root, we have

$$\overline{A} = 2mn^3(m^2+n^2)(\alpha^*)^2 + 2m^3n\alpha^* \qquad (31)$$

Substituting the values of $\overline{A}$ and $\overline{B}$ in (18),we get

$$a = (m^2+n^2)^2\left(2mn^3(m^2+n^2)(\alpha^*)^2 + 2m^3n\alpha^*\right)$$

$$b = (m^2+n^2)^2(2mn)^2 \frac{\alpha^*\left[(m^2+n^2)\alpha^*-1\right]}{2}$$

Using the values of a, b and α in (20),(16),(13) and (2), the corresponding integral solutions (1) are represented by

$$x = (m^2+n^2)^3(2mn\alpha^*)^2\left\{\begin{aligned} &\left((m^2+n^2)n^2\alpha^*+m^2\right)^2(2m^2) + \left((m^2+n^2)\alpha^*-1\right)^2(2m^2n^4) \\ &-(2mn)^2\left((m^2+n^2)n^2\alpha^*+m^2\right)\left((m^2+n^2)\alpha^*-1\right) \end{aligned}\right\}$$

$$y = -(m^2+n^2)^3(2mn\alpha^*)^2\left\{\begin{aligned} &\left((m^2+n^2)n^2\alpha^*+m^2\right)^2(2n^2) + \left((m^2+n^2)\alpha^*-1\right)^2(2m^4n^2) \\ &+(2mn)^2\left((m^2+n^2)n^2\alpha^*+m^2\right)\left((m^2+n^2)\alpha^*-1\right) \end{aligned}\right\}$$

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$$z=(m^2+n^2)^3(2mn\alpha^*)^2\left\{\begin{matrix}(m^2-n^2)\left[\left((m^2+n^2)n^2\alpha^*+m^2\right)^2-\left((m^2+n^2)\alpha^*-1\right)^2(mn)^2\right]\\ -(2mn)^2\left((m^2+n^2)n^2\alpha^*+m^2\right)\left((m^2+n^2)\alpha^*-1\right)\end{matrix}\right\}$$

$$w=(2mn)(m^2+n^2)^2\alpha^*$$

By considering the negative value for the square root on the right hand side of (21) another solution pattern is obtained.

For example,taking $m=2, n=1$ and following the analysis presented above the corresponding two sets of non-zero integral solutions to (1) are given by

Set (i)

$$x=5^3(200d^2)$$
$$y=-5^3(5000d^4)$$
$$z=5^3(100d^2-2500d^4)$$
$$w=50d$$

Set (ii)

$$x=5^3(20000d^4)$$
$$y=-5^4(10d^2)$$
$$z=5^3(10000d^4-25d^2)$$
$$w=50d$$

Conclusion

To conclude one may search for other pattern of solutions and their corresponding properties.

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